Do not use any unapproved aids while taking this assessment. Read each question carefully and be sure to show all work in the space provided.

1. Let V be the set of all pairs (x, y) of real numbers together with the following operations:

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 x_2,\ y_1 y_2)$$

$$c \odot (x, y) = (x^c,\ y^c)\,.$$

(a) Show that there exists an additive identity element, that is:

There exists $(w, z) \in V$ such that $(x, y) \oplus (w, z) = (x, y)$.

(b) Explain why V nonetheless is not a vector space.

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2. Consider each of these claims about a vector equation.

(a) "$\begin{bmatrix} -12 \\ 10 \\ -6 \\ -6 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -5 \\ 4 \\ -3 \\ -3 \end{bmatrix}$, and $\begin{bmatrix} 7 \\ -5 \\ 6 \\ 6 \end{bmatrix}$."

i. Write a statement involving the solutions of a vector equation that's equivalent to this claim.

ii. Determine if the statement you wrote is true or false.

iii. If your statement was true, describe a linear combination of $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -5 \\ 4 \\ -3 \\ -3 \end{bmatrix}$, and $\begin{bmatrix} 7 \\ -5 \\ 6 \\ 6 \end{bmatrix}$ that equals $\begin{bmatrix} -12 \\ 10 \\ -6 \\ -6 \end{bmatrix}$.

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3. (a) Write a statement involving the solutions of a vector equation that's equivalent to each claim:

i. "The set of vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ -4 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ -2 \end{bmatrix} \right\}$ **spans** $\mathbb{R}^4$."

ii. "The set of vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ -4 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ -2 \end{bmatrix} \right\}$ does **not** span $\mathbb{R}^4$."

(b) Explain how to determine which of these statements is true.

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4. Consider the following two sets of Euclidean vectors:

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, 2x + 5y = 5z \right\} \qquad W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, 3x^3 y + 4z = 0 \right\}$$

Explain why one of these sets is a subspace of $\mathbb{R}^3$ and one is not.

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5. (a) Write a statement involving the solutions of a vector equation that's equivalent to each claim:

i. "The set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 13 \\ -3 \\ 9 \\ 5 \end{bmatrix} \right\}$ is linearly **independent**."

ii. "The set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 13 \\ -3 \\ 9 \\ 5 \end{bmatrix} \right\}$ is linearly **dependent**."

(b) Explain how to determine which of these statements is true.

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6. (a) Write a statement involving the solutions of a vector equation that's equivalent to each claim:

i. "The set of vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a **basis** for $\mathbb{R}^4$."

ii. "The set of vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 2 \\ 3 \end{bmatrix} \right\}$ is **not** a basis for $\mathbb{R}^4$."

(b) Explain how to determine which of these statements is true.

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7. Consider the following subspace W of $\mathbb{R}^4$:

$$W = \text{span}\left\{ \begin{bmatrix} -5 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 15 \\ -9 \\ -9 \\ -9 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ -1 \\ -4 \end{bmatrix}, \begin{bmatrix} 15 \\ 1 \\ -7 \\ -1 \end{bmatrix} \right\}.$$

(a) Explain and demonstrate how to find a basis of W.

(b) Explain and demonstrate how to find the dimension of W.

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8. (a) Write a statement involving the solutions to a polynomial equation that's equivalent to each claim about the following set of polynomials:

$$\left\{-2\,x^3-2\,x^2+2\,x-1,-4\,x^3-4\,x^2+4\,x-2,x^3+4\,x^2-3\,x-1,4\,x^3+4\,x^2-4\,x+2,2\,x^3-x^2+3\right\}$$

i. "The set of polynomials **spans** $\mathcal{P}_3$. "

ii. "The set of polynomials does **not** span $\mathcal{P}_3$. "

(b) Explain how to determine which of these statements is true.

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9. Consider the following maps of polynomials $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ defined by

$$S(h(x)) = -x^2 - 3\,h\left(x^2\right) \quad \text{and} \quad T(h(x)) = -4\,h\,(5) + 3\,h'\,(x)$$

Explain why one these maps is a linear transformation and why the other map is not.

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10. (a) Find the standard matrix for the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 5\,x_1 - 8\,x_2 \\ 2\,x_1 - 3\,x_2 \\ 2\,x_1 + x_2 \\ -2\,x_2 \end{bmatrix}.$$

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} -1 & -1 \\ -2 & -3 \\ 3 & 6 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

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11. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + 3\,x_2 + 3\,x_3 + 12\,x_4 \\ -2\,x_1 + 6\,x_2 + 5\,x_3 + 21\,x_4 \\ -5\,x_3 - 15\,x_4 \end{bmatrix}.$$

(a) Explain and demonstrate how to find the image of T and the kernel of T.

(b) Explain and demonstrate how to find a basis of the image of T and a basis of the kernel of T.

(c) Explain and demonstrate how to the rank and nullity of T, and why the rank-nullity theorem holds for T.

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12. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $\begin{bmatrix} 1 & 2 & -4 & 8 \\ -1 & -1 & 0 & 1 \\ 0 & -1 & 5 & -12 \end{bmatrix}$.

(a) Explain and demonstrate why T is or is not injective.

(b) Explain and demonstrate why T is or is not surjective.

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13. Let A be a 4×4 matrix.

(a) Give a 4×4 matrix Q that may be used to perform the row operation $R_1 \leftrightarrow R_2$.

(b) Give a 4×4 matrix P that may be used to perform the row operation $R_2 + 5R_4 \to R_2$.

(c) Use matrix multiplication to describe the matrix obtained by applying $R_2 + 5R_4 \to R_2$ and then $R_1 \leftrightarrow R_2$ to A (note the order).

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14. Consider each of the following matrices.

(a)

$$L = \begin{bmatrix} -3 & 6 & 2 & 8 \\ -1 & 2 & 5 & 7 \\ 2 & -4 & -3 & -7 \\ 1 & -2 & -1 & -3 \end{bmatrix}$$

i. Explain why this matrix is or is not invertible by discussing its corresponding linear transformation.

ii. If the matrix is invertible, explain how to find its inverse.

(b)

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15. Let A be a 4×4 matrix with determinant -2.

(a) Let B be the matrix obtained from A by applying the row operation $R_4 \leftrightarrow R_1$. What is $\det B$?

(b) Let P be the matrix obtained from A by applying the row operation $-3R_2 \to R_2$. What is $\det P$?

(c) Let Q be the matrix obtained from A by applying the row operation $R_2 + 2R_3 \to R_2$. What is $\det Q$?

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16. Show how to compute the determinant of the matrix

$$A = \begin{bmatrix} -1 & 2 & 3 & 3 \\ -3 & 1 & 2 & 5 \\ 1 & 0 & 4 & 4 \\ 3 & 0 & 0 & -3 \end{bmatrix}.$$

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17. Explain and demonstrate how to find the eigenvalues of the matrix $\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$.

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18. Explain how to find a basis for the eigenspace associated to the eigenvalue -1 in the matrix

$$\begin{bmatrix} 0 & -3 & 3 & 2 \\ -2 & 5 & -6 & -4 \\ 3 & -9 & 8 & 6 \\ -5 & 15 & -15 & -11 \end{bmatrix}.$$